

Module 6: Hypothesis Testing with Sample Means

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Research and Statistical Support



Introduction to Statistics for the Social Sciences



The RSS short courses

The Research and Statistical Support (RSS) office at the University of North Texas hosts a number of “Short Courses”. A list of them is available at:

<http://www.unt.edu/rss/Instructional.htm>

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1 Sampling Distributions

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- 2 Z-test with Means Example

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- 3 Confidence Intervals

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- 4 Summary of Module 6

From a score to a distribution of scores

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 - Now we ask, does this sample of 3 cartoon dogs (Scooby, Pluto, & Goofy) differ from a particular population?

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 - We were given μ and σ , and compared Scooby's Z-score to the Z-critical value.
 - Now we ask, does this sample of 3 cartoon dogs (Scooby, Pluto, & Goofy) differ from a particular population?
 - Now, we are given μ and σ , and must compare a sample mean to the **Distribution of Means**.

Distribution of Means

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 - Also called: the Sampling Distribution of the Mean, as well as the Distribution of Sample Means.

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http://onlinestatbook.com/stat_sim/sampling_dist/index.html

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 - So, we correct for this bias.
- 3 The shape of the distribution of means is approximately normal if (a) each sample is of 30 or greater individuals, or (b) the distribution of the population is normal.

More on Rule 1

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 - Central Limit Theorem

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 - Even with large numbers of large samples, we may not have any single sample mean which is exactly equal to the population mean.
 - Larger sample sizes are always better than smaller sample sizes.

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 - Mathematical proof showing that no matter the shape of a population distribution, with an infinite number of random samples (with replacement), the distribution of sample means will be normally distributed around a mean which is equal to the population mean.

http://onlinestatbook.com/stat_sim/sampling_dist/index.html

More on Rule 2 (Part A)

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- This corrects for the known bias of the distribution of means' variability.
 - The variability of the distribution of means is always less than the population's variability.

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- **Rule 2b:** The standard deviation of a distribution of means is the square root of the variance of the distribution of means.

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- The standard deviation of a distribution of means is called the **Standard Error of the Mean (SEM)** or just the **Standard Error (SE)**.

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- Therefore, we can use what we know about the standard normal curve.
 - Percentage of scores, probabilities, and p values of Z-scores.

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$$Z = \frac{\bar{X} - \mu_M}{\sigma_M}$$

- Essentially, we're doing the same **Z-test** we were doing in the previous module, just modified slightly.

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We found that Scooby's Z-score was not more extreme than 1.64. Meaning, Scooby's IQ was not significantly greater than dogs not in cartoons.

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We found that Scooby's Z-score was not more extreme than 1.64. Meaning, Scooby's IQ was not significantly greater than dogs not in cartoons.

Now, we have a sample of 3 cartoon dogs (Scooby, Pluto, & Goofy). We are expanding our research to include a sample greater than one individual, but we are pursuing the same research question. Do cartoon dogs have significantly higher IQ than other dogs (i.e., those not on cartoons)?

The Sample Data

Table 1: Raw Data

X	Dog	IQ
1	Scooby	123
2	Pluto	145
3	Goofy	133

$$\frac{\sum X}{n} = \frac{401}{3} = 133.67$$

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Null Hypothesis: $H_0 : \mu_1 = \mu_2$

Alternative Hypothesis: $H_0 : \mu_1 > \mu_2$

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Standard deviation of the distribution of means:

$$\sigma_M = \sqrt{\sigma_M^2} = \sqrt{75} = 8.67$$

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We know from previous experience our cutoff score or critical value is a Z-score of 1.64

<http://www.sjsu.edu/faculty/gerstman/EpiInfo/z-table.htm>

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Clearly 3.88 is a very extreme Z-score.

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We have demonstrated that our sample is more extreme than our cutoff of 5% ($p = .05$) and conclude that; based on our (very small) sample, dogs on cartoons have a higher average IQ than dogs not on cartoons.

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- Interval estimates have gained popularity, in part because they tend to give us a better *idea* of where the population value is located.
 - Using a range of possible values that likely include the *unknown* population parameter.

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 - Dealing with distributions of means for most of the upcoming modules.
 - Standard deviation of a distribution of means is the **Standard Error (SE)**: σ_M
- Confidence Interval (CI).
 - Using the *SE*, we can calculate the range above and below the mean on a distribution of means that reflects the uncertainty associated with our sample.

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 - Typically 95% confidence interval.
 - 95% translates to +1.96 and -1.96 Z-scores above and below the mean on the standard normal curve.

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$$(+1.96)(8.67) + 133.67 = 150.6632$$

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- Lower Limit
$$(-Z\text{-cutoff})(SE) + \bar{X} = \text{lower limit}$$
$$(-1.96)(8.67) + 133.67 = 116.6768$$
- So, the 95% confidence interval from our sample is 150.66 to 116.68.

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- What we really want to know is the range that includes the population mean with $p = .95$ or we wish we could get a 95% chance that the interval includes the population mean.
 - Because we are dealing with a sample and a distribution of samples; we do not really know what the population value is and therefore, can not know what that interval would be.

Summary of Module 6

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Many of these topics will be revisited consistently in future modules.

This concludes Module 6

Next time Module 7.

- Next time we'll begin covering Additions to Statistical Significance.
- Until next time; have a nice day.

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